ANALYSIS III BACKPAPER EXAMINATION

Total marks: 100

(1) What is the equation of the tangent plane at any point (a, b, c) of the sphere $x^2 + y^2 + z^2 = 1$ in $\mathbb{R}^3$? (10 marks)

(2) Determine the points of $\mathbb{R}$ around which the sine function has a smooth local inverse function. What is the derivative of the inverse function (when it exists)? (10 marks)

(3) Let $f : [0, 1] \times [0, 1] \to \mathbb{R}$ be the function defined by $f(x, y) = \frac{1}{q}$ if y is rational and if $x = \frac{p}{q}$, where p, q are positive integers without common factors; let $f(x, y) = 0$ otherwise. Show that $\int_{[0,1]\times[0,1]} f$ exists. Verify Fubini's theorem for f. (10+10=20 marks)

(4) Compute the area of the torus described the parametrization:

$$r(u, v) = ((a + bcos(u))sin(v), (a + bcos(u))cos(v), bsin(u))$$

where $0 < b < a$ and $0 \leq u \leq 2\pi$, $0 \leq v \leq 2\pi$. (20 marks)

(5) Use Stokes theorem to prove $\int_C ydx + zdy + xdz = \pi a^2\sqrt{3}$, where C is the curve of intersection of the sphere $x^2 + y^2 + z^2 = a^2$ and the plane $x + y + z = 0$. Also explain how to traverse C to arrive at the given answer. (20 marks)

(6) Let V be the solid cylinder $x^2 + y^2 = 1$ and $-1 \leq z \leq 1$. Let S be the boundary of V, let n be the unit outward normal vector on points of S. Let $F(x, y, z) = (z, y, x)$. Verify Gauss's divergence theorem by computing both sides. (10+10=20 marks)